

Black-holes, topological strings and large N phase transitions

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Abstract. The counting of microstates of BPS black-holes on local Calabi-Yau of the form $\mathcal{O}(p-2) \oplus \mathcal{O}(-p) \longrightarrow S^2$ is explored by computing the partition function of q -deformed Yang-Mills theory on S^2 . We obtain, at finite N , the instanton expansion of the gauge theory. It can be written exactly as the partition function for $U(N)$ Chern-Simons gauge theory on a Lens space, summed over all non-trivial vacua, plus a tower of non-perturbative instanton contributions. In the large N limit we find a peculiar phase structure in the model. At weak string coupling the theory reduces to the trivial sector and the topological string partition function on the resolved conifold is reproduced in this regime. At a certain critical point, instantons are enhanced and the theory undergoes a phase transition into a strong-coupling regime. The transition from the strong-coupling phase to the weak-coupling phase is of third order.

1. Introduction

A novel and highly non trivial relation between the topological string vacuum amplitude Z_{top} and the partition function Z_{BH} of $N = 2$ BPS black-holes in four dimensions has been recently suggested in [1]: $Z_{\text{BH}} = |Z_{\text{top}}|^2$. This intriguing equality generalizes a series of beautiful results [2, 3] concerning the entropy of BPS black holes arising in compactifications of Type II superstrings on Calabi-Yau threefolds and it is supposed to hold for a large black hole charge N at any order in the $\frac{1}{N}$ expansion, once taken into account the *perturbative* definition of Z_{top} . Actually the proposal of [1] is even more startling, since it suggests to employ the above relation

as a non perturbative definition of the topological string dynamics.

In order to check this proposal, one should engineer Calabi-Yau backgrounds in which both sides of the relation can be computed independently. For this task, in [4], generalizing the original example presented in [5], the following class of non-compact Calabi-Yau threefolds,

$$\mathcal{M} = \mathcal{O}(p + 2g - 2) \oplus \mathcal{O}(-p) \longrightarrow \Sigma_g , \quad (1)$$

has been considered. The manifold Σ_g is a Riemann surface of genus g , while $\mathcal{O}(m)$ is a holomorphic line bundle of degree m over Σ_g . The counting of BPS states on these geometries has been claimed to reduce to computing the partition function of a peculiar deformation of Yang-Mills theory on Σ_g called q -deformed Yang-Mills. Starting from this result one should ask if the relation with the perturbative topological string amplitudes holds in this case. Happily the partition function Z_{top} for these geometries has been computed very recently [6] and the consistency check amounts to reproducing these amplitudes as the large N limit of q -deformed Yang-Mills theory on Σ_g . In [4] a large N expansion has been performed and the conjecture was confirmed, but with a couple of important subtleties. Firstly, one should include in the definition of Z_{top} a sum over a $U(1)$ degree of freedom identified with a Ramond-Ramond flux through the Riemann surface. Secondly, and more importantly, the relevant topological string partition function implies the presence of $|2g - 2|$ stacks of D-branes inserted in the fibers of \mathcal{M} . An explanation of this unexpected feature in terms of extra closed string moduli, related to the non-compactness of \mathcal{M} , has been offered in [7]. Recent works on this topic include [8]–[14].

That two-dimensional Yang-Mills theory should be related to a string theory in the large N limit is not entirely unexpected due to the well-known Gross-Taylor expansion [15]–[18]. At large N , the partition function of two-dimensional Yang-Mills theory on a Riemann surface Σ_g almost factorizes into two copies (called chiral and antichiral) of the same theory of unfolded branched covering maps, the target space being Σ_g itself. The chiral-antichiral factorization is, however, violated by some geometrical structures called orientation-reversing tubes.

The emergence of chiral and antichiral sectors was also observed in [4] in studying the q -deformed version of two dimensional Yang-Mills theory and it implies the appearance of the modulus squared $|Z_{\text{top}}|^2$, a crucial ingredient in checking the relation with black hole physics. However, on S^2 , this picture may be jeopardized by the q -deformed incarnation of the Douglas-Kazakov phase transition occurring at large N . There, a strong-coupling phase, wherein the theory admits the Gross-Taylor string description, is separated by a weak-coupling phase with gaussian field theoretical behaviour. Instanton configurations induce the transition to strong-coupling [18], while the entropy of branch points appears to be responsible for the divergence of the string expansion above the critical point [19, 20].

Here we shall explore in detail q -deformed Yang-Mills theory on S^2 elucidating its relation with topological string theory on the threefold $\mathcal{M} = \mathcal{O}(p - 2) \oplus \mathcal{O}(-p) \rightarrow \mathbb{P}^1$ and its rich phase diagram. The details of this analysis can be found in [21] and related work in [22, 23].

2. Black holes, topological strings and q-deformed Yang-Mills theory

2.1. The conjecture

The conjecture presented in [1] is more easily phrased in the context of Type IIA superstring theory compactified on $\mathcal{M} \times \mathbb{R}^{3,1}$, where \mathcal{M} is a Calabi-Yau threefold. In this framework a BPS black hole can be realized by wrapping D6, D4, D2 and D0 branes around holomorphic cycles in \mathcal{M} . The D2 and D0-brane charges are referred to as “electric”, while D6 and D4-brane charges are “magnetic”. One can define a partition function for a mixed ensemble of BPS black hole states by fixing the magnetic charges Q_6 and Q_4 and summing over the D2 and D0 charges with fixed chemical potentials ϕ_2 and ϕ_0 to get

$$Z_{\text{BH}}(Q_6, Q_4, \phi_2, \phi_0) = \sum_{Q_2, Q_0} \Omega(Q_6, Q_4, Q_2, Q_0) \exp[-Q_2 \phi_2 - Q_0 \phi_0] , \quad (2)$$

where $\Omega(Q_6, Q_4, Q_2, Q_0)$ is the *number* of BPS states with fixed D-brane charges. The conjecture relates the partition function (2) to the (A-model) topological string partition function $Z_{\text{top}}(g_s, t_s)$ on \mathcal{M}

$$Z_{\text{BH}}(Q_6, Q_4, \phi_2, \phi_0) = |Z_{\text{top}}(g_s, t_s)|^2 , \quad (3)$$

where the topological string coupling g_s and the Kähler modulus t_s are related to the black-hole charges as follows

$$g_s = \frac{4\pi i}{\frac{i}{\pi} \phi_0 + Q_6} , \quad t_s = \frac{1}{2} g_s \left(-\frac{i}{\pi} \phi_2 + N Q_4 \right) \quad (4)$$

The origin of this proposal can be traced back to some well-known properties of BPS black holes in $\mathcal{N} = 2$ supergravity [2, 3]. These are solutions interpolating between two maximally supersymmetric vacua: the Minkowski space at infinity and the Bertotti geometry in the near horizon region. Moreover they carry charges (P^I, Q_I) with the respect to $n_v + 1$ abelian gauge fields present in the theory: the n_v matter vector multiplets and the graviphoton. The entropy for this class of geometries is determined by the Bertotti mass M_{Bert} appearing in the near horizon geometry and, in turn, M_{Bert} is a function of the scalar fields X^I of the vector multiplets at the horizon. This potential dependence on the scalars is problematic, because it contrasts with the idea that the black-hole entropy is a universal quantity, fixed by the conserved charges (*no-hair theorem*). What comes to rescue is the so-called *attractor mechanism* [25, 26]. The scalar fields of the theory X^I have fixed values at the black hole horizon determined only by the charges, through the geometric equations¹

$$P^I = \text{Re}(X^I) = \oint_{A_I} \text{Re}(\Omega) \quad \text{and} \quad Q_I = \text{Re}(F_I) = \oint_{B^I} \text{Re}(\Omega). \quad (5)$$

Here Ω is the holomorphic three-form on the Calabi-Yau \mathcal{M} while (A_I, B^I) are a basis of $H_3(\mathcal{M}, \mathbb{Z})$. The period F_I is the gradient of the prepotential $F_0(= X^I F_I)$. By means of the explicit expressions in eq. (5) one obtains $S_{\text{BH}} \simeq Q_I X^I - P^I F_I$. The entropy thus appears as

¹ To be precise, this supergravity analysis is phrased in the mirror setting of type IIB.

a Legendre transform of the prepotential F_0 , which coincides with the genus zero free energy of topological strings in the Calabi-Yau background. At the supergravity level, one can include higher-derivative corrections proportional to $R^2 T^{2g-2}$, where T is the graviphoton field-strength, and recompute the black hole solutions and their entropies [2, 3]. At genus $g = 1$, the relation with the topological string free energy still holds when one includes quantum corrections to the prepotential coming from one-loop amplitudes [27, 28]. The suggestion of [1] is obviously consistent with these results and cleverly suggests a generalization of them to all orders in the perturbative string expansion. It is of course natural to attempt to check this conjecture on some explicit Calabi-Yau threefold \mathcal{M} . While for the compact case the task seems out of reach presently, in the non-compact case there is the general class of threefolds (1) on which the problem has been attacked [4, 5]. The study of topological strings on these backgrounds and the related counting of microstates have also produced a number of interesting mathematical results. As we will see, different gauge theories in diverse dimensions appear to be related by their common gravitational ancestor.

2.2. Counting microstates in $\mathcal{N} = 4$ and q -deformed Yang-Mills theories

Let us begin by describing the counting of microstates. It consists of counting bound states of D4, D2 and D0-branes, where N D4 branes wrap the four-cycle C_4 , which is the total space of the holomorphic line bundle $\mathcal{O}(-p) \rightarrow \Sigma_g$, and the D2-branes wrap the Riemann surface Σ_g . The natural way of doing the computation is by studying the relevant gauge theory on the brane. According to the general framework [29] the worldvolume gauge theory on the N D4-branes is the $\mathcal{N} = 4$ topologically twisted $U(N)$ Yang-Mills theory on C_4 . The presence of chemical potentials for D2 and D0-branes is simulated by turning on the observables in the theory given by

$$S_c = \frac{1}{2g_s} \int_{C_4} \text{Tr}(F \wedge F) + \frac{\theta}{g_s} \int_{C_4} \text{Tr}(F \wedge K) , \quad (6)$$

where F is the Yang-Mills field strength and K is the unit volume form of Σ_g . The relation between the gauge parameters g_s , θ and the chemical potentials are $\phi_0 = 4\pi^2/g_s$ and $\phi_2 = 2\pi\theta/g_s$ in accordance with the identifications for the D0 and D2-brane charges q_0 , q_2 as

$$q_0 = \frac{1}{8\pi^2} \int_{C_4} \text{Tr}(F \wedge F) , \quad q_2 = \frac{1}{2\pi} \int_{C_4} \text{Tr}(F \wedge K) . \quad (7)$$

Evaluating Z_{BH} is thus equivalent to computing the expectation value in topologically twisted $\mathcal{N} = 4$ Yang-Mills theory given by

$$Z_{\text{BH}} = \left\langle \exp \left[-\frac{1}{2g_s} \int_{C_4} \text{Tr}(F \wedge F) - \frac{\theta}{g_s} \int_{C_4} \text{Tr}(F \wedge K) \right] \right\rangle = Z_{\mathcal{N}=4} . \quad (8)$$

The general structure of this functional integral has been explored in [30]. There it was shown that the partition function $Z_{\mathcal{N}=4}$ has an expansion of the form

$$Z_{\mathcal{N}=4} = \sum_{q_0, q_2} \Omega(q_0, q_2; N) \exp \left(-\frac{4\pi^2}{g_s} q_0 - \frac{2\pi\theta}{g_s} q_2 \right) , \quad (9)$$

where $\Omega(q_0, q_2; N)$ is the Euler characteristic of the moduli space of $U(N)$ instantons on C_4 in the topological sector labelled by the zeroth and second Chern numbers q_0 and q_2 . The counting of microstates is therefore equivalent to an instanton counting in the $\mathcal{N} = 4$ topological gauge theory. This is still a formidable problem, because no general strategy exists in the case of non-compact manifolds and very few results [31, 32, 33] have been obtained in this context.

However in [5] the computation was related to a two-dimensional problem. In fact by introducing certain massive deformations and under the reasonable assumption that the path integral localizes to $U(1)$ -invariant modes around the fiber $\mathcal{O}(-p)$, it was argued that the theory reduces to an effective gauge theory on Σ_g . In [5] it was also shown that the non-triviality of the line bundle $\mathcal{O}(-p)$ generates an extra term in the effective two-dimensional action of the form

$$S_p = -\frac{p}{2g_s} \int_{\Sigma_g} \text{Tr} \Phi^2 K \quad (10)$$

where $\Phi(z)$ parameterizes the holonomy of the gauge field A around a circle at infinity in the fiber over the point $z \in \Sigma_g$. The relevant two-dimensional action becomes

$$S_{\text{YM}_2} = \frac{1}{g_s} \int_{\Sigma_g} \text{Tr}(\Phi F) + \frac{\theta}{g_s} \int_{\Sigma_g} \text{Tr} \Phi K - \frac{p}{2g_s} \int_{\Sigma_g} \text{Tr} \Phi^2 K. \quad (11)$$

This is just the action of two-dimensional Yang-Mills theory on the Riemann surface Σ_g . However there is an important subtlety. The new degree of freedom Φ is periodic due to its origin as the holonomy of the gauge field at infinity and this periodicity affects the path integral measure in a well-defined way [4]. The final result is

$$Z_{\mathcal{N}=4} = Z_{\text{YM}}^q = \sum_R \dim_q(R)^{2-2g} q^{\frac{p}{2} C_2(R)} e^{i\theta C_1(R)} \quad \text{with} \quad \dim_q(R) = \prod_{1 \leq i < j \leq N} \frac{[R_i - R_j + j - i]_q}{[j - i]_q}. \quad (12)$$

This is to be compared with the partition function of ordinary Yang-Mills theory on Σ_g given by the Migdal expansion [35]

$$Z_{\text{YM}} = \sum_R \dim(R)^{2-2g} e^{\frac{g^2 A}{2} C_2(R)} e^{i\theta C_1(R)} \quad \text{with} \quad \dim(R) = \prod_{1 \leq i < j \leq N} \frac{R_i - R_j + j - i}{j - i}, \quad (13)$$

where R runs through the unitary irreducible representations of the gauge group $U(N)$, $\dim(R)$ is its dimension, and $C_1(R)$ and $C_2(R)$ are respectively its first and second Casimir invariants. The dimensionless combination $g^2 A$ of the Yang-Mills coupling constant g^2 and area A of Σ_g is the effective coupling of the theory, as dictated by invariance under area-preserving diffeomorphisms. The main effect of the deformation of the path integral measure is to turn the ordinary dimension $\dim R$ into the *quantum* dimension $\dim_q R$. The deformation parameter q is related to the coupling g_s through $q = e^{-g_s}$ and the deformed integer number $[n]_q \equiv (q^{n/2} - q^{-n/2}) / (q^{1/2} - q^{-1/2})$. Clearly as $g_s \rightarrow 0$ the quantum dimension goes smoothly into the classical one. The effective dimensionless coupling of the gauge theory is $g^2 A = g_s p$.

Eq. (12) provides the solution of the difficult problem of instanton counting, as it represents the answer on all four-manifolds C_4 . The direct comparison with eq. (9) seems at first sight puzzling, because the expansion of the q -deformed gauge theory is in e^{-g_s} and not in e^{-1/g_s} . This implies that a modular transformation is required in eq. (12) and we will discuss this point in Sect. 3.

2.3. Large N limit and topological strings

Eq. (12) gives, in principle, the exact expression for the black hole partition function Z_{BH} . In order to establish the connection with the perturbative topological string partition function on \mathcal{M} , we need to take the limit of large charges, in other words the large N limit of the q -deformed Yang-Mills theory. It is important, first of all, to set the relation between the parameters of the gauge theory describing the black hole and the data of the closed topological string theory. For the Kähler modulus t_s of the base Σ_g , with the help of the attractor mechanism, one finds

$$t_s = 2\pi i \frac{X^1}{X^0} = (p + 2g - 2) \frac{Ng_s}{2} - i\theta. \quad (14)$$

The closed topological string emerging from the large N limit is thereby expected to possess the modulus (14). This is the first very non-trivial feature that should be reproduced by the large N limit. The second non-trivial point is the square modulus structure. This is not completely unexpected due to the relation with the large N limit of ordinary Yang-Mills theory. In the Gross-Taylor expansion [16] two types of representations contribute to the limit, called the chiral representations (with much less than N Young tableaux boxes) and the antichiral representations (with order N boxes). The partition function is almost factorized into two copies (apart from subtleties). One could wonder whether Z_{YM}^q factorizes as well. The choice of relevant representations is dictated by the Casimir dependence of the partition function, which is unchanged by the deformation. We expect $Z_{\text{YM}}^q \simeq Z_{\text{YM}}^{q,+} Z_{\text{YM}}^{q,-}$. Moreover, one would expect that the chiral q -deformed Yang-Mills partition function $Z_{\text{YM}}^{q,+}$ could be written as a holomorphic function of t_s and identified with the topological string amplitude $Z_{\text{top}}(g_s, t_s)$ on \mathcal{M} . In [4] these expectations have been confirmed, but with some subtleties as well.

For $g = 0$, the case studied extensively in this proceeding, the result is

$$Z_{\text{YM}}^q(S^2) = \sum_{l=-\infty}^{\infty} \sum_{\hat{R}_1, \hat{R}_2} Z_{\hat{R}_1, \hat{R}_2}^{q\text{YM},+}(t_s + p g_s l) Z_{\hat{R}_1, \hat{R}_2}^{q\text{YM},-}(\bar{t}_s - p g_s l) \quad (15)$$

with $Z_{\hat{R}_1, \hat{R}_2}^{q\text{YM},-}(\bar{t}_s) = (-1)^{|\hat{R}_1|+|\hat{R}_2|} Z_{\hat{R}_1^\top, \hat{R}_2^\top}^{q\text{YM},+}(\bar{t}_s)$. Here \hat{R}_i are irreducible representations of $SU(N)$ and the chiral block $Z_{\hat{R}_1, \hat{R}_2}^{q\text{YM},+}(t_s)$ agrees exactly with the perturbative topological string amplitude on \mathcal{M} [6] with 2 stacks of D-branes inserted in the fiber. Here $|\hat{R}|$ is the total number of boxes of the Young tableau of the representation \hat{R} . The chiral and anti-chiral parts are sewn along the D-branes and summed over them. The extra sum over the integer l originates from the $U(1)$ degrees of freedom contained in the original gauge group $U(N)$. The generalization to higher genus follows the same path, but it is slightly different [4].

The genus zero case is also special because it admits a standard description in terms of toric geometry. The fibration $X = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \rightarrow \mathbb{P}^1$ is in fact a toric manifold [24] and the partition function can be written in terms of the topological vertex $C_{\hat{R}_1 \hat{R}_2 \hat{R}_3}(q)$ [36] as

$$Z_{\hat{R}_1, \hat{R}_2}^{q\text{YM},+}(t_s) = Z_0 q^{k_{\hat{R}_1}/2} e^{-\frac{t_s(|\hat{R}_1|+|\hat{R}_2|)}{p-2}} \sum_{\hat{R}} e^{-t_s|\hat{R}|} q^{(p-1)k_{\hat{R}_1}/2} C_{0\hat{R}_1\hat{R}^\top}(q) C_{0\hat{R}\hat{R}_2}(q), \quad (16)$$

where $k_{\hat{R}}$ is related to the Young tableaux labels through $k_{\hat{R}} = \sum_i \hat{R}_i(\hat{R}_i - 2i + 1)$ and Z_0 represents the contribution from constant maps. This is the partition function of the topological A-model on \mathcal{M} with non-compact D-branes inserted at two of the four lines in the web diagram. The D-branes are placed at a well-defined “distance” $\hat{t} = t_s/(p-2)$ from the Riemann surface, thereby introducing another geometrical modulus.

We thus observe an apparent discrepancy between the prediction of [1] that $Z_{\text{BH}} = |Z_{\text{top}}|^2$ and the explicit computation leading to $Z_{\text{BH}} = \sum_{b,l} |Z_{\text{top}}^{(b,l)}|^2$, where index b denotes the sum over chiral blocks with branes inserted. However, the extra sum over the integer l has been naturally interpreted as a sum over R-R fluxes through Σ_g [5]. The sum over the fiber D-branes is instead related to the fact that a non-compact Calabi-Yau may have additional moduli coming from the non-compact directions [7]. This “external” sum is weighted with a different Kähler parameter $\hat{t} = t_s/(p-2)$, and the partition function effectively depends on two parameters. This suggests that \hat{t} could be interpreted as a new Kähler modulus [24].

The whole picture therefore seems very convincing and pointing towards a beautiful confirmation of the conjecture presented in [1]. However, there is a point that has been overlooked which could have interesting ramifications. In taking the large N limit it is possible to encounter phase transitions. The prototype of this kind of phenomenon was discovered long ago [37, 38] in the one-plaquette model of lattice gauge theory.

A well-known large N phase transition is the Douglas-Kazakov transition [17]. It concerns Yang-Mills theory on the sphere, a close relative of the relevant black hole ensembles discussed earlier. A strong-coupling phase, in which the theory admits the Gross-Taylor string description with its chiral-antichiral behaviour, is separated by a weak-coupling phase with gaussian field theoretical behavior. Two-dimensional Yang-Mills theory on S^2 is equivalent to a string theory only above a certain critical value of the effective t’Hooft coupling constant $\lambda = Ng^2A$ given by $\lambda_c = \pi^2$. Instanton configurations induce the phase transition to the strong-coupling regime [18]. On the other hand, the entropy associated to certain classes of branched covering maps seems responsible for the divergence of the string perturbation series above the critical point [19, 20]. It is natural to expect that, if q -deformed large N Yang-Mills theory is related to the undeformed one, some of these features could find a place in the black holes/topological string scenario. We will explore this possibility in the subsequent sections.

3. Large N -limit of q -deformed YM_2 : one-cut solution and weak-coupling regime

We discuss now the large N limit of q -deformed Yang-Mills theory on the sphere. An explicit result for the leading order (planar) contribution to the free energy of ordinary Yang-Mills theory on the sphere was obtained in [17]. For large area it fits nicely with the interpretation in terms of branched coverings that arises in the Gross-Taylor expansion, down to the phase transition point at $\lambda_c = \pi^2$ where the string series is divergent. We will now perform similar computations for q -deformed Yang-Mills theory on S^2 . We start by defining the relevant parameters to be held fixed as $N \rightarrow \infty$ by $t = g_s N$ and by $a = g_s p N = p t$. The partition function of the q -deformed gauge theory on S^2 is given by

$$Z_{\text{YM}}^q(g_s, p) = \sum_{\substack{n_1, \dots, n_N \in \mathbb{Z} \\ n_i - n_j \geq i - j \text{ for } i \geq j}} e^{-\frac{q_s p}{2}(n_1^2 + \dots + n_N^2)} \prod_{1 \leq i < j \leq N} \sinh^2\left(\frac{q_s}{2}(n_i - n_j)\right), \quad (17)$$

where we have chosen $\theta = 0$. The constraint on the sums keeps track of the meaning of the integers n_i in terms of Young tableaux labels and highest weights. In terms of these new variables, the partition function at $N \rightarrow \infty$ is given by $Z_{\text{YM}}^q(t, a) = \exp(N^2 S_{\text{eff}}(\rho))$ where

$$S_{\text{eff}} = - \int_C dw \int_C dw' \rho(w) \rho(w') \log \left[\sinh\left(\frac{t}{2}|w - w'|\right) \right] + \frac{a}{2} \int_C dw \rho(w) w^2. \quad (18)$$

We have introduced the variable $x_i = i/N$ and the function $n(x)$ such that $n(x_i) = \frac{n_i}{N}$. In the large N limit, x_i becomes a continuous variable $x \in [0, 1]$: the density ρ is defined as $\rho(n) = \frac{\partial x(n)}{\partial n}$ and we denoted the interval $[n(0), n(1)]$ by C . The distribution $\rho(z)$ in eq. (18) can be determined by requiring that it minimizes the action. This implies that it satisfies the saddle-point equation

$$\frac{a}{2} z = \frac{t}{2} \int_C dw \rho(w) \coth\left(\frac{t}{2}(z - w)\right). \quad (19)$$

This equation is a deformation of the usual Douglas-Kazakov equation that governs ordinary QCD₂ on the sphere. The ordinary gauge theory is recovered when $t \rightarrow 0$ while a is kept fixed. The one-cut solution of eq. (19) is given in [44] and its explicit form is given by

$$\rho(z) = \frac{a}{\pi t} \arctan \left(\sqrt{\frac{e^{t^2/a}}{\cosh^2(\frac{t}{2}z)} - 1} \right) \quad (20)$$

with the symmetric support $z \in [-\frac{2}{t} \text{arccosh}(e^{-t^2/2a}), \frac{2}{t} \text{arccosh}(e^{-t^2/2a})]$. It can be considered as the q -deformation of the well-known Wigner semi-circle distribution. We now come to the crucial point: in the continuum limit the constraint on the series (17) becomes $n(x) - n(y) \geq x - y$ for $x \geq y$: we may translate the original constraint in terms of the function ρ as $\rho(n) \leq 1$ [17]. The above bound is of fundamental importance. In fact its violation signals a potential large N phase transition with the consequent existence of a strong-coupling phase. In terms of the variables t and p , the bound on ρ produces the inequality

$$\arctan \left(\sqrt{e^{t/p} - 1} \right) \leq \frac{\pi}{p}, \quad (21)$$

a condition that is always satisfied for $p = 1$ or $p = 2$. The situation changes for $p > 2$ and the inequality (21) can be equivalently written as $\sqrt{e^{t/p} - 1} \leq \tan \frac{\pi}{p}$ which implies that

$$t \leq t_c = p \log \left(\sec^2 \left(\frac{\pi}{p} \right) \right) . \quad (22)$$

Our solution, therefore, breaks down when the 't Hooft coupling t reaches the critical value t_c : we remark that the cases $p = 1, 2$ are special because then the one-cut solution is always valid. The breakdown of the one-cut solution parallels exactly what happens in ordinary two-dimensional Yang-Mills theory, where it signals the appearance of a phase transition. In the saddle-point approach it is possible to go further and to find a solution describing the strong-coupling phase, for $t > t_c$. Before proceeding with this analysis, we compute the free energy in the weak-coupling phase and we discuss its topological string interpretation. A tedious but elementary integration gives

$$\mathcal{F}(t, a) = -\frac{t^2}{6a} + \frac{\pi^2 a}{6t^2} - \frac{a^2}{t^4} \zeta(3) + \frac{a^2}{t^4} \text{Li}_3(e^{-t^2/a}) + c(t) . \quad (23)$$

The a -independent function $c(t)$ can be easily determined by looking at the asymptotic expansion in t . In the limit $t \rightarrow 0$, $p \rightarrow \infty$ the free-energy of ordinary Yang-Mills theory in the weak-coupling phase is of course recovered. We observe that the free energy depends only on the combination t^2/a (or $t_s/(p(p-2))$ in string variable). This dependence is in contrast with the one expected from the geometry of the Calabi-Yau $\mathcal{O}(p-2) \oplus \mathcal{O}(-p) \rightarrow S^2$. This picture, in fact, contains as a necessary ingredient two independent moduli e^{-t_s} and $e^{-\frac{2t_s}{p-2}}$: we miss the Calabi-Yau geometry (there is no substantial dependence on p , that simply scales the only Kähler modulus without affecting the structure of the free-energy) and the modulus square. It appears clear that in the weak-coupling regime we cannot reproduce the geometrical structure predicted by the conjecture: while for $p > 2$ we could expect that above the critical point a strong-coupling solution will do it, we conclude that this is impossible for $p = 1, 2$. Actually the free-energy eq.(23) coincides with the genus 0 free-energy of closed topological string theory on the resolved conifold. Thus in the weak-coupling phase we have $Z_{\text{qYM}} = Z_{\text{top}}^{\text{Res.}}$ with Kähler modulus t^2/a . The loss of the original geometric information encoded in q -deformed YM_2 and the mysterious appearance of the resolved conifold can be easily understood by studying the large N limit of the gauge theory in the dual instanton picture.

The geometrical meaning of the q -deformed theory on S^2 becomes more transparent when we consider the dual description in terms of instantons, provided by a modular transformation of the series (17). This is accomplished by means of a Poisson resummation. It is also an efficient way to investigate the behaviour of the theory at weak-coupling g_s , as we shall explain soon. By exploiting the properties of the Stieltjes-Wigert orthogonal polynomials, we can easily obtain the instanton expansion of q -deformed Yang-Mills theory on S^2

$$Z_{\text{YM}}^q(g_s, p) = \frac{1}{N!} \sum_{s_i \in \mathbb{Z}} e^{-\frac{2\pi^2}{g_s p} \sum_{i=1}^N s_i^2} w_q^{\text{inst}}(s_1, \dots, s_N) \quad (24)$$

where

$$w_q^{\text{inst}}(s_1, \dots, s_N) = \frac{1}{2} \left(\frac{2\pi}{g_s p} \right)^N e^{-\frac{g_s(N^3 - N)}{6p}} \int_{-\infty}^{\infty} dz_1 \cdots dz_N e^{-\frac{2\pi^2}{g_s p} \sum_{i=1}^N z_i^2} \\ \times \prod_{1 \leq i < j \leq N} \left[\cos \left(\frac{2\pi(s_i - s_j)}{p} \right) - \cos \left(\frac{2\pi(z_i - z_j)}{p} \right) \right]. \quad (25)$$

Our terminology mimicks that of the undeformed theory where the partition function can be computed exactly via a nonabelian localization [34]. It is given by a sum over contributions localized at the classical solutions of the theory. For finite N the $U(N)$ path integral is given by a sum over unstable instantons where each instanton contribution is given by a finite, but non-trivial, perturbative expansion. By “instantons” we mean solutions of the classical Yang-Mills field equations, which are not gauge transformations of the trivial solution $A = 0$. On the sphere S^2 , the most general solution is given by $(A(z))_{ij} = \delta_{ij} A^{(m_i)}(z)$ where $A^{(m_i)}(z)$ is the Dirac monopole potential of magnetic charge m_i . The Yang-Mills action evaluated on such an instanton is given by $S_{\text{inst}} = \frac{2\pi^2}{g^2 A} \sum_{i=1}^N m_i^2$. Poisson resummation exactly provides the representation of ordinary Yang-Mills theory on S^2 in terms of instantons [18, 40]. Looking closer at eqs. (24) and (25) we recognize a similar structure emerging. We observe the expected exponential of the “classical action” $e^{-\frac{2\pi^2}{g_s p} \sum_{i=1}^N s_i^2}$ and the fluctuations $w_q^{\text{inst}}(s_1, \dots, s_N)$ which smoothly reduce to the undeformed ones in the double scaling limit. The instanton representation is also useful to control the asymptotic behaviour of the partition function as $g_s \rightarrow 0$. In this limit, only the zero-instanton sector survives, the others being exponentially suppressed (for fixed p).

We immediately recognize that all the non-trivial instanton contributions are nonperturbative in the $\frac{1}{N}$ expansion, being naively exponentially suppressed, suggesting that the theory could reduce in this limit to the zero-instanton sector. In order for this possibility to be correct, one should control the fluctuation factors. In ordinary Yang-Mills theory, the corrections due to the contribution of instantons to the free energy were calculated in [18]. There, it was found that while in the weak-coupling phase this contribution is exponentially small, it blows up as the phase transition point is approached. The transition occurs when the entropy of instantons starts dominating over their Boltzmann weight $e^{-S_{\text{inst}}}$.

In principle, the q -deformed theory on S^2 could experience the same fate. In eq. (24) the Boltzmann weights are the same as in the undeformed case, and only the structure of the fluctuations is changed by the deformation. One way to detect the presence of a phase transition at $N \rightarrow \infty$ is to look for a region in the parameter space where the one-instanton contribution dominates the zero-instanton sector [18]. In our case the ratio of the two contributions is given by (we have dropped an irrelevant normalization factor)

$$e^{F_0} = \frac{\int \prod_{i=1}^N dz_i e^{-\frac{8\pi^2 N}{a} \sum_{i=1}^N z_i^2} \prod_{j=2}^N [\sin^2(\frac{2\pi t}{a}(z_1 - z_j)) - \sin^2(\frac{\pi t}{a})] \prod_{\substack{i < j \\ i \geq 2}} \sin^2(\frac{2\pi t}{a}(z_i - z_j))}{e^{\frac{2\pi^2 N}{a}} \int \prod_{i=1}^N dz_i e^{-\frac{8\pi^2 N}{a} \sum_{i=1}^N z_i^2} \prod_{1 \leq i < j \leq N} \sin^2(\frac{2\pi t}{a}(z_i - z_j))}. \quad (26)$$

We have employed the saddle-point technique to compute the above ratio and we recovered the same results of the one-cut analysis: at $t = t_c$ one-instanton contributions are no longer suppressed and zero-instanton approximation breaks down. We do not enter into the details of this computation but we find instructive to see explicitly how the behaviour changes above the value $p = 2$ by plotting the integral defining F_0 (Fig. 1). The appearing of topological

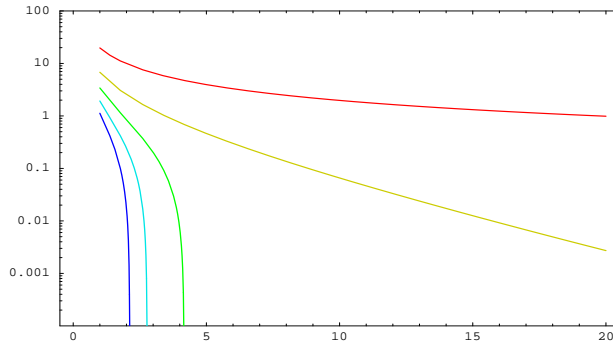


Figure 1. Plotting F_0 for $p = 1, 2, 3, 4, 5$ as a function of t

string amplitudes on the resolved conifold in the large N limit can be explained as follows. A remarkable property of the fluctuations is that they do not depend really on the integers s_i but only their values modulo p . It is natural to organize the partition function by factorizing the independent fluctuations. We can then write down the partition function in the suggestive form

$$Z_{YM}^q = \sum_{\{N_k\}} \prod_{k=0}^{p-1} (N_k!)^{-1} \theta_3 \left(\frac{2\pi i p}{g_s} \middle| \frac{2\pi i k}{g_s} \right)^{N_k} Z_{CS}^p(\{N_k\}), \quad (27)$$

where we recognize in the second line the partition function of $U(N)$ Chern-Simons gauge theory on the Lens space L_p in a non-trivial vacuum given by [41, 42]

$$Z_{CS}^p(\{N_k\}) = \exp \left(-\frac{2\pi^2}{g_s p} \sum_{m=0}^{p-1} N_m m^2 \right) w_q^{\text{inst}}(\underbrace{0, \dots, 0}_{N_0}, \dots, \underbrace{p-1, \dots, p-1}_{N_{p-1}}). \quad (28)$$

The relation we have found should be understood as an analytical continuation to imaginary values of the Chern-Simons coupling constant k by identifying $g_s p = \frac{2\pi i}{k+N}$. The critical points of the $U(N)$ Chern-Simons action on the manifold L_p are flat connections which are classified by the embeddings of the first fundamental group into $U(N)$: one has $\pi_1(L_p) = \mathbb{Z}_p$. The critical points are therefore given by discrete \mathbb{Z}_p -valued flat connections. They are easily described by choosing N -component vectors with entries taking values in \mathbb{Z}_p . Because the residual Weyl symmetry S_N of the $U(N)$ gauge group permutes the different components, the independent choices are in correspondence with the partitions $\{N_k\}$. The possible vacua of the gauge theory are in one-to-one correspondence with the choices of flat connections. The full partition function

of Chern-Simons theory involves summing over all the flat connections, and in fact the exact answer that can be obtained from the relation with the WZW model [43] gives such a sum. Nevertheless, due to the fact that the flat connections here are isolated points, it is not difficult to extract the particular contribution of a given vacuum which coincides with eq. (28). Let us consider the large N limit of the zero-instanton sector. Inserting $w_q^{\text{inst}}(0, \dots, 0)$ into the instanton expansion we obtain

$$\mathcal{Z}^{0-\text{inst}} = \frac{1}{N!} \left(\frac{2\pi}{g_s p} \right)^N e^{-\frac{g_s(N^3-N)}{6p}} \int \prod_{i=1}^N dz_i e^{-\frac{8\pi^2 N}{a} \sum_{i=1}^N z_i^2} \prod_{1 \leq i < j \leq N} \sin^2 \left(\frac{2\pi t(z_i - z_j)}{a} \right) . \quad (29)$$

The partition function (29) coincides with the partition function Z_{CS}^p of Chern-Simons theory on L_p in the background of the trivial flat-connection with partition $\{N_k\} = (0, \dots, 0)$. According to [41, 42], eq. (29) is its matrix-model representation. The large N limit in the trivial vacuum can be explicitly performed by using the orthogonal polynomial technique explained in [44] or it can be obtained from the same result for S^3 in [45] by simply identifying the parameters. The closed topological string theory on the resolved conifold $\mathcal{O}(1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ emerges in the limit as an effect of the geometric transition.

4. Two-cut solution and the strong-coupling phase

For $t > t_c$ we still have to solve the saddle point equation, but in the presence of the boundary condition $\rho \leq 1$. The new feature which may arise is that a finite fraction of Young tableaux variables n_1, \dots, n_N condense at the boundary of the inequality by respecting the parity symmetry of the problem, $n_{k+1} = n_{k+2} = \dots = n_{N-k} = 0$, while all others are non-zero. This observation translates into a simple choice for the profile of $\rho(z)$ as depicted in Fig. 2. In order

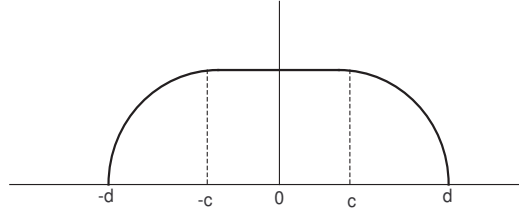


Figure 2. The double cut ansatz for the distribution $\rho(z)$.

to respect the bound, the distribution function is chosen constant and equal to 1 everywhere in the interval $[-c, c]$. In the intervals $[-d, -c]$ and $[c, d]$ its form is instead dynamically determined by the saddle-point equation. Let us denote the set $[-d, -c] \cup [c, d]$ by \mathcal{R} and the restriction of the distribution function to this set by $\tilde{\rho} = \rho|_{\mathcal{R}}$. The saddle-point equation can be rewritten as

$$\frac{a}{2} z - \log \left| \frac{\sinh \left(\frac{t}{2} (z + c) \right)}{\sinh \left(\frac{t}{2} (z - c) \right)} \right| = \frac{t}{2} \int_{\mathcal{R}} dw \tilde{\rho}(w) \coth \left(\frac{t}{2} (z - w) \right) . \quad (30)$$

Changing variables $y = e^{tz} e^{t^2/a}$ and $u = e^{tw} e^{t^2/a}$, we get:

$$\frac{a}{2t^2} \frac{\log(y)}{y} - \frac{1}{ty} \log \left(\frac{e^{ct-t^2/a} y - 1}{e^{-ct-t^2/a} y - 1} \right) = \int_{\mathcal{R}} du \frac{\hat{\rho}(u)}{y - u}. \quad (31)$$

We have defined $\hat{\rho}(u) = \tilde{\rho}(\log(u e^{-t^2/a})/t)/ut$ and the support is now given by $\mathcal{R} = [\underbrace{e^{-td+t^2/a}}_{e^{d_-}}, \underbrace{e^{-tc+t^2/a}}_{e^{c_-}}] \cup [\underbrace{e^{tc+t^2/a}}_{e^{c_+}}, \underbrace{e^{td+t^2/a}}_{e^{d_+}}]$. The original potential has been non-trivially modified by a term that depends explicitly on the endpoints. The saddle point solution is determined by the two following equations:

$$\begin{aligned} \frac{a}{2t} \int_0^\infty dw \log(w) \frac{d}{dw} \left(\frac{1}{\sqrt{(w + e^{c_-})(w + e^{c_+})(w + e^{d_+})(w + e^{d_-})}} \right) \\ = \int_{e^{c_-}}^{e^{c_+}} \frac{dw}{w} \frac{1}{\sqrt{(w - e^{c_-})(e^{c_+} - w)(e^{d_+} - w)(w - e^{d_-})}}, \end{aligned} \quad (32)$$

$$\begin{aligned} \frac{a}{2t} \int_0^\infty \frac{dw}{\sqrt{(w + e^{c_-})(w + e^{c_+})(w + e^{d_+})(w + e^{d_-})}} \\ = \int_{e^{c_-}}^{e^{c_+}} \frac{dw}{\sqrt{(w - e^{c_-})(e^{c_+} - w)(e^{d_+} - w)(w - e^{d_-})}}. \end{aligned} \quad (33)$$

Eqs. (32,33) are very complicated and do not seem promising for an analytical treatment: by introducing $k = \sqrt{\frac{(e^{c_+} - e^{c_-})(e^{d_+} - e^{d_-})}{(e^{d_+} - e^{c_-})(e^{c_+} - e^{d_-})}}$ they can be nevertheless expressed in terms of elliptic functions. The situation then improves dramatically if we change variable from k to the modular parameter $q = \exp(-\pi K'(k)/K(k))$, $K(k)$ denoting the complete elliptic integral of the first type. In particular the problem becomes equivalent to solve a single equation, expressed through an elegant q -series given by

$$\frac{t}{4} = \frac{t_c}{4} - 2p \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{q^{2n}}{1 - q^2} \sin^2 \left(\frac{\pi n}{p} \right) = -\frac{p}{2} \log \left(\frac{\vartheta_2(\frac{\pi}{p}|q)}{\vartheta_2(0|q)} \right) \equiv \mathfrak{F}(p, q). \quad (34)$$

The graphical behaviour of the function on the right-hand side of eq. (34) is depicted in Fig. 3, and one can check that it is a monotonically increasing function as k runs from 0 to 1: $\frac{t}{4}$ must be greater than the value of the right-hand side of eq. (34) at $k = 0$, obtaining the expected bound $t \geq 4\mathfrak{F}(p, 0) = t_c$. At the critical point t_c , we have $q = 0$: the two cuts merge and form a single cut whose endpoints coincide with those of the one-cut solution. To go further and see what happens in a neighborhood around t_c we have to solve eq. (34) as a series in $(t - t_c)$. We assume that q admits an expansion of the form $q = (t - t_c)^\alpha \sum_{n=0}^{\infty} a_n (t - t_c)^n$. Substituting this expansion into eq. (34), we can iteratively solve for the coefficients a_n and find (here $\hat{t} = t - t_c$)

$$q = \sqrt{\hat{t}} \left(\frac{\csc(\frac{\pi}{p})}{\sqrt{8p}} + \frac{\hat{t} \cos(\frac{2\pi}{p}) \csc^3(\frac{\pi}{p})}{32\sqrt{2}p^{3/2}} + \frac{\hat{t}^2 \left(5 \cos(\frac{4\pi}{p}) - 32 \cos(\frac{2\pi}{p}) - 27 \right) \csc^5(\frac{\pi}{p})}{6144\sqrt{2}p^{5/2}} + O(\hat{t}^3) \right). \quad (35)$$

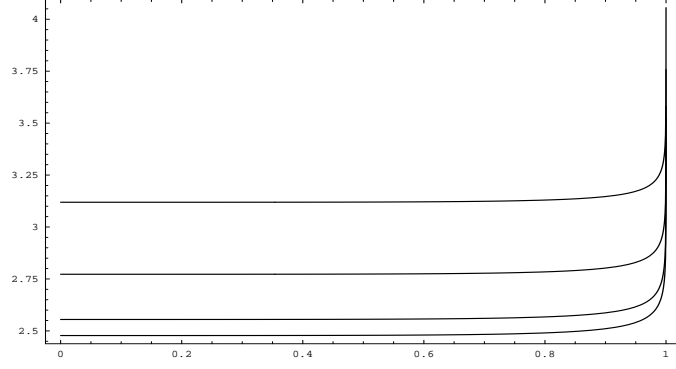


Figure 3. The right-hand side of the saddle-point equation times p for $p = 3, 4, 7, 20$.

From this expansion we can obtain all information about the gauge theory in the strong-coupling phase around the critical point. To investigate the behaviour of the theory beyond the phase transition, we need to understand what happens to the distribution function ρ . Because our potential is non-polynomial, we have no simple relations relating derivatives of the free energy to the expansion of the resolvent, as in the standard matrix models or in ordinary Yang-Mills theory. We have to resort therefore to a brute force calculation. The second derivative of the free energy can be reduced to computing

$$\frac{\partial^2 \mathcal{F}}{\partial a^2} \propto \frac{1}{2} \int_{e^{c_-}}^{e^{d_+}} dz \frac{\partial \rho(z)}{\partial a} \left(\log(z) - \frac{t^2}{a} \right)^2 + \frac{1}{2} \int_{e^{d_-}}^{e^{c_-}} dz \frac{\partial \rho(z)}{\partial a} \left(\log(z) - \frac{t^2}{a} \right)^2 \quad (36)$$

as all other contributions vanish because of the boundary conditions on the distribution and its symmetries. The derivatives are taken at constant t . A tedious expansion of this quantity around $t = t_c$ using *Mathematica* shows that it vanishes linearly in $(a - a_c)$ and thus the phase transition is of third order. We can also present some evidences for how the topological string expansion emerges. The topological string perturbative expansion is naturally organized as a double series in two modular parameters e^{-t_s} and $e^{-2t_s/(p-2)}$, where t_s is the Kähler modulus (related to our t by $t = \frac{2t_s}{p-2}$). The appearance of this double dependence from our equation is non-trivial and absolutely necessary for the relevant string interpretation. Because we expect that the topological string theory will arise when t is large, we have to investigate the solution of our saddle-point equation around $t = \infty$: k and consequently q approach 1 and it is natural to perform a modular transformation on our equation. This procedure exchanges τ and $-\frac{1}{\tau}$, and thus a perturbative solution can be attempted. The saddle-point equation then becomes

$$\frac{t}{4} = -\frac{p}{2} \log \left(\frac{\vartheta_2\left(\frac{\pi}{p}|q\right)}{\vartheta_2(0|q)} \right) = -\frac{\tilde{\tau}}{2p} - \frac{p}{2} \log \left(\frac{\vartheta_4\left(\frac{i\tilde{\tau}}{p}|\tilde{q}\right)}{\vartheta_4(0|\tilde{q})} \right) \quad (37)$$

where we have defined $\tau = iK'/K \equiv \pi/i\tilde{\tau}$ and $\tilde{q} = e^{\pi i\tilde{\tau}}$. At leading order in the solution we have $\tilde{\tau} = -\frac{tp}{2} = -t_s - \frac{2t_s}{p-2}$. The corrections coming from the theta-functions are exponentially

suppressed at this level. To explore the subleading order we proceed iteratively and we get

$$\tilde{\tau} = -t_s - \frac{2t_s}{p-2} + 4p^2 \sum_{n=1}^{\infty} \frac{e^{-nt_s - \frac{2n t_s}{p-2}}}{1 - e^{-2nt_s - \frac{4n t_s}{p-2}}} \sinh^2 \left(\frac{n t_s}{p-2} \right), \quad (38)$$

and so on. It is evident that the solution for the modular parameter $\tilde{\tau}$ and thus the partition function nicely organizes into a double expansion in the two moduli e^{-t_s} and $e^{-2t_s/(p-2)}$ as expected from string theory.

Acknowledgments

D.S. warmly thanks the organizers for the kind invitation at the meeting and the hospitality.

This list of references is very far from being complete because of the limited space and we deeply apologize for that. A complete set of reference can be found in [21, 24]

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